# A MODEL OF ANOMALOUS ELASTIC NONLINEARITY OF MICROINHOMOGENEOUS MEDIA

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**Abstract**

*A simple model is proposed and analyzed to explain anomalously high elastic nonlinearity typical of solids with microinhomogeneous inner structure, such as polycrystalline solids, grainy media, crack-containing solids, etc.*

## Introduction.

At present, there is a rather wide variety of so-called microinhomogeneous media which demonstrate anomalously high and qualitatively unusual acoustic nonlinearity [1–4]. The inner structure of such media is characterized by the presence of various inhomogeneities and defects whose size is large compared with the inter-atom distance, but is small with respect to the characteristic scale of the acoustic perturbation.

From the viewpoint of acoustics, the use of the 'classical' theory of nonlinear elasticity is often insufficient to explain the anomalous nonlinear behaviour of these media. It is realized now that the anomalous nonlinearity should be attributed to the influence of structure inhomogeneities absent in ideal solids, in which nonlinearity is associated with anharmonicity of the inter-atomic potential, which may be quite satisfactorily described by conventional 5-constant elasticity theory [2,4]. Creation of a model of micro-structure-induced nonlinearity requires special consideration of each particular type of medium inhomogeneities, and that is, generally speaking, a rather complex problem. There are a few examples of such physical models, though only the simplest cases allow for a relatively complete and consistent analysis. These physical models of structure-induced elastic nonlinearity are practically constrained by simplified considerations of granular [4, 5], rubber-like porous [6, 7], and crack-containing [8, 9] elastic media. The conclusions derived from each of these models can be applied to the corresponding particular kind of microinhomogeneous material, but are of limited use for generalization and application to other types of medium defect.

An alternative approach, based on purely phenomenological reconstruction of a medium 'stress-strain' constitutive law can be applied for description of nonlinear effects in media with rather different kinds of inner (micro)structure. Successful examples of the use of such an approach can be found in [10, 11]. However, as the price for its wide working area, the phenomenological description does not allow us to conclude anything about the inner features of the medium described.

Below, a simple model of elastic nonlinearity is proposed, which may be characterized as an intermediate case between the above-mentioned physical models applied to special types of media and the models based on a purely phenomenological structure-independent description. In the framework of the proposed approach, the elastic nonlinear properties of the model microinhomogeneous medium are determined and the role of structure inhomogeneities in the increase of the elastic nonlinearity is pointed out. The relation of the obtained results to real microinhomogeneous media is also discussed.

## The medium model

Let us take as a base for consideration a popular model of one-dimensional elastic media that is constructed as a chain of equal masses $M$ and elastic elements-springs which for simplicity's sake are supposed to have equal lengths $L$. The typical wavelength $\Lambda$ of elastic perturbations that we shall consider is supposed to be large, $\Lambda \gg L$.

The principal point of the model is that there are a certain number of very soft springs (modeling the medium defects), the elastic coefficient $\chi_1$ of which is much less than the elastic coefficient $\chi$ of other springs, $\chi/\chi_1 \equiv Q \gg 1$. The linear density of the chain elements is characterized by the value $N$, while the density of the soft inclusions is equal to $N_1$. Then let us suppose for convenience that the cross-section $S$ of each elastic element has the unit square, so the values $N$ and $N_1$ may be considered as volume densities, and the elastic force acting at each element corresponds to the elastic stress $\sigma$ in the medium.

In the long-wave (quasi-static) approximation the value of the elastic stress $\sigma$ is evidently equal at each element:

$$\sigma = \chi X = \chi_1 X_1 \ , \tag{1}$$

and, therefore, the corresponding deformations $X_1$ of the soft and rigid elements $X$ are different :

$$X_1 = (\chi/\chi_1)\, X \equiv QX \gg X \ . \tag{2}$$

For the total deformation $X^t$ of the chain consisting of $N$ elements we obtain, using the above-introduced definitions:

$$X^t = \sum_1^{N-N_1} X + \sum_1^{N_1} X_1 \,. \tag{3}$$

Substitution of (2) into (3) yields the following relation between the total chain deformation and the deformation $X$ of each rigid spring:

$$X^t = XN\left[1+(N_1/N)\,(Q-1)\right] \ . \tag{4}$$

It follows from (4) that if all of the springs have equal rigidity (i.e. $Q=1$), the total deformation $X^t = XN$, as it should be in the defectless chain. Then, dividing equation (4) by the initial chain length $N$, the mean chain strain and the strain of each rigid spring $X/L$ may be related:

$$\varepsilon = X/L\,\left[1+\nu\,(Q-1)\right] \ , \tag{5}$$

where $\nu = N_1/N$ is the relative density of the soft springs. Therefore it follows from (5) that in the case of small density of the soft spring-defects ($\nu Q \ll 1$), their presence does not practically affect the mean strain, whose value appears to be very close to the strain of the defectless chain and, correspondingly, to each rigid spring strain.

To consider nonlinear corrections to the 'stress-strain' relation of the defect-containing media it is necessary to specify the nonlinear elastic properties of each elastic element. Let us suppose that the material of each spring is characterized by a small deviation from the linear Hooke's law: $\sigma = M\,\varepsilon\,(1+f(\varepsilon))$, where $f(\varepsilon) \ll 1$, $M = \chi/L$. As a rather general case we shall use a power-type nonlinearity law $f(\varepsilon) = \Gamma^{(n)}\,\varepsilon^{n-1}$, where $\Gamma^{(n)}$ is a dimensionless nonlinearity coefficient of the $n$-th order, $n > 1$. Therefore for the material of the rigid springs we suppose that

$$\sigma = M\varepsilon\,(1+\varepsilon^{n-1}\,\Gamma^{(n)}) \ , \tag{6}$$

and for the soft spring material :

$$\sigma = M_1\,\varepsilon\,(1+\varepsilon^{n-1}\,\Gamma^{(n)}) \ , \tag{7}$$

where the linear elastic moduli are related as $M/M_1 = \chi/\chi_1 \equiv Q$. We suppose deliberately that the nonlinearity coefficients of the material of both types of spring are of equal 'normal' (that is about several units, $\Gamma^{(n)} \sim 10^{0}$) value. However, as will be shown in the following section, the high contrast in just the linear elastic properties may lead to anomalous growth of the mean elastic nonlinearity of the defected chain.

### Nonlinear elastic properties of the defected chain.

To determine the nonlinearity of the defect-containing chain we shall start from the relation of the mean stress $\sigma$ and the mean density of the elastic energy $W$:

$$\sigma = \partial W/\partial\varepsilon \ . \tag{8}$$

The mean energy density $W$ is related to the elastic energy $W^r$ and $W^s$ stored correspondingly at each of the rigid and soft springs by the following evident expression:

$$W = (N - N_1)W^r + N_1 W^s, \tag{9}$$

where the individual energies are determined by the deformations of the springs $X$ and $X_1$:

$$W^r = \int_0^X M\,[\xi/L + \Gamma^{(n)}\xi^n/L^n]\,d\xi \tag{10}$$

$$W^s = \int_0^{X_1} M_1\,[\xi/L + \Gamma^{(n)}\xi^n/L^n]\,d\xi\,. \tag{11}$$

Using relations (2), (8)−(11), after straightforward derivation we obtain the nonlinear 'stress-strain' relation for the inhomogeneous medium:

$$\sigma = \varepsilon \frac{M}{(1-\nu+\nu Q)}\left\{1 + \varepsilon^{n-1}\Gamma^{(n)}\frac{1-\nu+\nu Q^n}{(1-\nu+\nu Q)^n}\right\}. \tag{12}$$

In the limit case of the homogeneous chain when $Q = 1$ or $\nu = 0$, equation (12) coincides with equation (6). Therefore the mean elastic properties of the homogeneous chain naturally coincide with those of each elastic element.

Let us consider now the dependence of linear and nonlinear elastic properties upon the density $0 < \nu < 1$ of the soft ($Q \gg 1$) springs. It follows from equation (12) that the mean nonlinearity coefficient $\Gamma^{(n)}_{mean}$ is equal to $\alpha\Gamma^{(n)}$, where

$$\alpha = \frac{1 + \nu(Q^n - 1)}{[1 + \nu(Q-1)]^n}, \tag{13}$$

and the mean elastic modulus $M_{mean}$ is equal to $\beta M$, where

$$\beta = [1 + \nu(Q-1)]^{-1}. \tag{14}$$

It is evident from (13), (14) that, when the defect density is very small $\nu Q^n \ll 1$, both linear and nonlinear mean coefficients are practically unchanged $\alpha = \beta = 1$.

When $\nu Q \ll 1$, but $\nu Q^n \gg 1$, the mean nonlinear coefficient increases significantly ($\alpha \approx \nu Q^n \gg 1$), unlike the linear modulus which still remains practically the same: $\beta \approx 1$.

With further increase of the density $\nu$, when $\nu Q \gg 1$ but $\nu \ll 1$, the mean rigidity of the defect-containing chain begins to decrease ($\beta \approx (\nu Q)^{-1} \ll 1$), but the mean value of the nonlinear coefficient of the chain remains anomalously high ($\alpha \approx \nu^{1-n} \gg 1$, as the exponent $n > 1$).

At last, when $\nu \to 1$ the elastic modulus of the chain becomes equal to the elastic modulus of the soft springs ($\beta = Q^{-1}$, therefore $M_{mean} = M/Q$), and the nonlinear coefficient returns to its normal value ($\alpha = 1$).

The maximal value of $\Gamma^{(n)}$ is reached at the optimal defect density

$$\nu_{opt} = \frac{1}{n-1}\frac{1}{Q-1} - \frac{n}{n-1}\frac{1}{Q^n - 1} \approx \frac{1}{(n-1)Q} \ll 1, \tag{15}$$

the corresponding value of the increase factor $\alpha(\nu_{opt})$ of the mean nonlinearity coefficient being equal to:

$$\alpha(\nu_{opt}) \approx \frac{(n-1)^{n-1}}{n^n} Q^{n-1} \gg 1, \tag{16}$$

while the relative variation of the linear elastic modulus is much smaller:

$$\beta(\nu_{opt}) \approx \frac{(n-1)}{n} \sim 1. \tag{17}$$

**Conclusion.**

Let us summarize the results of the above consideration of variability of mean linear and nonlinear properties of microinhomogeneous (defect-containing) media.

The model gives a simple qualitative explanation for the anomalously high nonlinearity that has been observed in different experiments and has even been considered sometimes as an experimental mistake, being in contradiction with traditional views based on 'classical' nonlinear elasticity of homogeneous solids.

Equations (12)−(17) show that even a very small density of soft ($Q \ll 1$) defects may cause drastic variations of the medium nonlinearity (by several orders of magnitude) while its linear acoustic characteristics may remain insignificantly perturbed. This statement is corroborated by numerous experiments (see, e.g., [5, 10, 11, 12]) and is of great importance for possible diagnostic applications of nonlinear effects.

The higher-order nonlinear parameters (for example, the cubic nonlinearity $\Gamma^{(3)}$ compared with the quadratic parameter $\Gamma^{(2)}$ are influenced by the defects considerably more significantly, as follows from (13)−(16). This statement also agrees well with experimental data (see, e.g., [5, 12]).

Though the considered simple model does not correspond directly to a particular real microinhomogeneous medium, it offers clear heuristic guidelines for the analysis of concrete cases, as in each case one may easily recognize the model 'soft springs' in real defects such as pores, cracks, inter-grain contacts, etc. The ways of complication of the model are quite evident and may be considered elsewhere, though its simplicity and clarity seem to be the main merits of the above discussed refined variant.

Note lastly that the above consideration was restricted to the analysis of the nonlinear elastic properties themselves. Further, starting from the derived mean stress-strain relation (12) one may readily obtain the nonlinear wave equation in the long-wave approximation

$$\rho \partial^2 u/\partial t^2 = \partial^2 \sigma/\partial x^2 , \tag{18}$$

where $\rho = M/L$. Then equation (18) may be analyzed by conventional methods [2].

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